METRIC AND TOPOLOGICAL SPACES: EXAM 2021/22

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Problem 1 (15%). For all $x, y \in \mathcal{X}$ with a metric d put $\varrho(x,y) = \mathrm{d}(x,y)/(1+\mathrm{d}(x,y))$ by definition. Prove that the function $\varrho\colon \mathcal{X}\times\mathcal{X} \to [0,1)$ is another metric on $\mathcal{X}$.

Problem 2 (15%). Let a continuous map $f\colon \mathcal{X} \to \mathcal{Y}$ be a bijection of metric spaces; suppose $\mathcal{X}$ is compact. Prove that the inverse f^{-1} of bijection f is also continuous. (Hint: $\mathcal{Y}$ is Hausdorff.)

Problem 3 (15+10%). Prove a relation between boundaries:

$$\partial(A \cup B) \subseteq \partial A \cup \partial B, \qquad \text{here } A, B \subseteq \mathcal{X}.$$

- Give an example of indexed collection of sets such that $Y := \partial(\bigcup_{i\in I} A_i) \not\subseteq \bigcup_{i\in I} \partial A_i =: Z$ but the intersection $Y \cap Z = \varnothing$.

Problem 4 (15%). Let $\mathcal{X}$ be a space such that every continuous function $f\colon \mathcal{X} \to \mathbb{E}^1$ has the following property: if $a < c < b$, $f(x) = a$, and $f(y) = b$, then there exists $z \in \mathcal{X}$ such that $f(z) = c$. Prove $\mathcal{X}$ is connected.

Problem 5 (15%). Prove that on the set $[0,1] \subset \mathbb{R}$ there is no Hausdorff topology $\mathcal{T}_1$ which is strictly coarser ($\subsetneq$) than the Euclidean one, $\mathcal{T}_2$.

Problem 6 (15%). Solve for $x(s)$ the integral equation,

$$x(s) = \tfrac{1}{2}\int_0^1 s\cdot t\, x(t)\,\mathrm{d}t + \tfrac{5}{6}s,$$

by consecutive approximations starting from $x_0(s) = 0$. (In the end, verify by direct substitution that the function $x(s)$ which you have found satisfies the equation.)

Date: November 12, 2021. Good luck & take care !